Effective field theory for Sp(N) antiferromagnets and its phase structure

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In this paper, we study quantum Sp(N) antiferromagnetic (AF) Heisenberg models in two dimensions (2D) by using the Schwinger-boson representation and the path-integral methods. An effective field theory, which is an extension of CP^{N-1} model in (2+1)D, is derived and its phase structure is studied by the $1/N$-expansion. We introduce a spatial anisotropy in the exchange couplings and show that the effective coupling constant in the CP^{N-1} model is an increasing function of the anisotropy. For the SU(N) AF Heisenberg model, which is a specific case of the Sp(N) model, we found that phase transition from the ordered "Néel state" to paramagnetic phase takes place as the anisotropy is increased. In the vicinity of the SU(N) symmetric point, this phase structure is retained. However as a parameter that controls explicit breaking of the SU(N) symmetry is increased, a new phase, which is similar to the spiral-spin phase with a nematic order in frustrated SU(2) spin systems, appears. It is shown that at that phase transition point, a local SU(2) gauge symmetry with composite SU(2) gauge field appears in the low-energy sector. It is another example of symmetry-enhancement phenomenon at low energies. We also introduce a lattice gauge-theoretical model, which is a counterpart of the effective field theory, and study its phase structure by means of the Monte-Carlo simulations.

I. INTRODUCTION

Study on quantum antiferromagnets has been one of the most active area in the condensed matter physics. In particular in the last decade, much attention has been paid to exotic phase and exotic phase transition for which Landau's classic paradigm cannot be applicable[1]. It is expected that investigation of such exotic states is useful to understand anomalous properties of under doped high-T_c materials[2]. Furthermore recent development in technologies of ultracold atoms and optical lattice trap elevates purely academic quantum spin models to

realistic ones, and these cold-atom systems are sometimes regarded as a final simulator for strongly-correlated electron systems. Quantum SU(N) antiferromagnets are one of these examples. Theoretically these models can be studied by using the Schwinger-boson methods and the $1/N$-expansion[3]. In a seminal paper[4], it was shown that spin-$(N-1)/2$ (N is an even integer) cold atom systems in an optical lattice with one atom per quantum well can be regarded as quantum Sp(N) antiferromagnets. There are two parameters J_1, J_2 in the Hamiltonian of the Sp(N) magnets and when $J_1 = J_2$, the symmetry is enhanced to SU(N)⊃Sp(N) and a SU(N) quantum antiferromagnet is realized.

In the present paper, we shall study Sp(N) quantum antiferromagnets by using the slave-boson (Schwinger boson) representation and the path-integral methods. We first derive an effective field theory for the Sp(N) Heisenberg model. This field theory is an extension of the CP^{N-1} model for the SU(N) antiferromagnetic (AF) Heisenberg model. Then we investigate its phase structure by using the $1/N$-expansion. We also study numerically its lattice-gauge-model counterpart by means of the Monte-Carlo (MC) simulations.

This paper is organized as follows. In Sect.2, we shall derive the effective field theory for the Sp(N) AF Heisenberg model by using CP^{N-1} representation of (pseudo-)spin degrees of freedom. In Sect.3, we study the field-theory model by the $1/N$-expansion. We focus on quantum phase transition in the spatial two-dimensional (2D) system. We first show that in the SU(N) AF magnets with anisotropic exchange couplings on a square lattice, a phase transition from the AF Néel state to the paramagnetic state takes place as the anisotropy is increased. These two states persist in the Sp(N) system in the vicinity of the SU(N) symmetric point $J_1 = J_2$. As J_2/J_1 is decreased to some critical value, a phase transition to a new phase with a composite vector-field condensation takes place and a new kind of spin order appears. In Sect.4, we study a lattice version of the obtained field theory and show the results of the numerical study for the Sp(4) case. Obtained phase diagram is qualitatively in agreement with that obtained by the $1/N$-expansion, but we also find some discrepancy between the $1/N$-expansion and the numerical study, e.g. order of the phase transition, etc. Section 5 is devoted for conclusion. In the Appendix, some details of the derivation of the effective field theory are given.

II. MODEL AND EFFECTIVE FIELD THEORY

A. Sp(N) Heisenberg model

We consider anisotropic Sp(N) Heisenberg model on a square lattice, i.e., exchange couplings in the x and y-directions are different. Quantum Hamiltonian of the system is given as follows in the most general form,

$$\mathcal{H} = \sum_{\langle i,j \rangle} \left\{ J_1^{i,j} \sum_{a,b} \Gamma_i^{ab} \Gamma_j^{ab} - J_2^{i,j} \sum_a \Gamma_i^a \Gamma_j^a \right\}, \tag{2.1}$$

where i, j denote lattice sites and $\Gamma^{ab} \in \mathbf{sp}(N)$ (the Lie algebra of Sp(N)) have $\frac{N(N+1)}{2}$ components. On the other hand, $\Gamma^a \in \mathbf{su}(N)/\mathbf{sp}(N)$, and there are $\frac{(N+1)(N-2)}{2}$ components. Hereafter in most of cases, we set the exchange coupling $J_1^{i,j}$ and $J_2^{i,j}$ as follows,

$$J_n^{i,i+\hat{x}} = J_{n,x}, \ J_n^{i,i+\hat{y}} = J_{n,y}, \ (n=1,2), \ \text{otherwise } 0,$$

where $\hat{x}$ ($\hat{y}$) is the unit vector of the $x(y)$-direction. For the $N = 4$ case[5], the above generators are explicitly given as

$$\Gamma^a = \sigma^a \otimes \mu^z, a = 1, 2, 3, \ \ \Gamma^4 = 1 \otimes \mu^x, \ \ \Gamma^5 = 1 \otimes \mu^y, \tag{2.2}$$

and

$$\Gamma^{ab} = \frac{1}{2i}[\Gamma^a, \Gamma^b], \tag{2.3}$$

where $\sigma^a, \mu^a (a = 1, 2, 3)$ are two sets of the 2×2 Pauli matrices. The Sp(4) case corresponds to the spin-3/2 system, and Γ^a involves even powers of the SU(2) spin matrices, whereas Γ^{ab} involves odd powers of the spin matrices. Then the long-range order of Γ^a indicates a spin-nematic order.

It is useful to introduce the matrix $\mathcal{J}$, which has the following properties and is a generalization of the time-reversal matrix $i\sigma^2$ in the SU(2) spin case,

$$\mathcal{J}^t = -\mathcal{J}, \ \mathcal{J}^2 = -1, \ \mathcal{J}\Gamma^{ab}\mathcal{J} = (\Gamma^{ab})^t, \ \mathcal{J}\Gamma^a\mathcal{J} = -(\Gamma^a)^t. \tag{2.4}$$

For the Sp(4) case with Eqs.(2.2), $\mathcal{J} = i\sigma^y \otimes \mu^x$.

We introduce the Schwinger boson operator $\hat{b}_\alpha$ ($\alpha = 1, \cdots, N$) and represent the "spin operators" in the Hamiltonian (2.1) in terms of them, $\hat{\Gamma}_i^{ab} = \hat{b}_{i,\alpha}^\dagger \Gamma_{\alpha\beta}^{ab} \hat{b}_{i,\beta}, \ \ \hat{\Gamma}_i^a = \hat{b}_{i,\alpha}^\dagger \Gamma_{\alpha\beta}^a \hat{b}_{i,\beta}.$

By using the identities,

$$\sum_{a,b} \Gamma^{ab}_{\alpha\beta}\Gamma^{ab}_{\gamma\sigma} = 2\delta_{\alpha\sigma}\delta_{\beta\gamma} - 2\mathcal{J}_{\alpha\gamma}\mathcal{J}_{\beta\sigma}, \tag{2.5}$$

$$\sum_{a} \Gamma^{a}_{\alpha\beta}\Gamma^{a}_{\gamma\sigma} = 2\delta_{\alpha\sigma}\delta_{\beta\gamma} + 2\mathcal{J}_{\alpha\gamma}\mathcal{J}_{\beta\sigma} - \delta_{\alpha\beta}\delta_{\gamma\sigma}, \tag{2.6}$$

we obtain

$$\mathcal{H} = \sum_{\langle i,j\rangle}\{2(J_1^{i,j} - J_2^{i,j})\hat{K}_{ij}^\dagger\hat{K}_{ij} - 2(J_1^{i,j} + J_2^{i,j})\hat{Q}_{ij}^\dagger\hat{Q}_{ij}\}, \tag{2.7}$$

where $\hat{Q}_{ij} = \mathcal{J}_{\alpha\beta}\hat{b}_{i,\alpha}\hat{b}_{j,\beta}$ and $\hat{K}_{ij} = \hat{b}_{\alpha,i}^\dagger\hat{b}_{\alpha,j}$. The operator $\mathcal{J}_{\alpha\beta}\hat{b}_{j,\beta}$ is the conjugate spinor of $\hat{b}_{j,\beta}$ and then $\hat{Q}_{ij}$ represents pairing of spins on the lattice sites i,j, whereas $\hat{K}_{ij}$ corresponds to the Schwinger boson (spinon) hopping. Subsidiary condition

$$\sum_{\alpha=1}^{N} \hat{b}_{i,\alpha}^\dagger\hat{b}_{i,\alpha}|\text{phys}\rangle = |\text{phys}\rangle, \tag{2.8}$$

must be imposed as the physical-state condition. We redefine the exchange couplings as $J_{i,j} \equiv 2(J_1^{i,j} + J_2^{i,j})$, $J'_{i,j} \equiv 2(J_1^{i,j} - J_2^{i,j})$, then

$$\mathcal{H} = \sum_{\langle i,j\rangle}\{J'_{i,j}\hat{K}_{ij}^\dagger\hat{K}_{ij} - J_{i,j}\hat{Q}_{ij}^\dagger\hat{Q}_{ij}\}. \tag{2.9}$$

From Eq.(2.9), it is obvious that when $J'_{i,j} = 0$ ($J_{i,j} = 0$), the model have the global SU(N) symmetry. In the following section, we shall derive the effective field theory for the system (2.9) by using the path-integral methods.

B. Effective field theory

In this subsection, we shall derive the low-energy effective field theory of the Hamiltonian (2.9). To this end, we use the coherent path-integral methods. For the Schwinger boson, we use the CP^{N-1} boson that satisfies $\bar{z}_i \cdot z_i = \sum_\alpha \bar{z}_{i,\alpha}z_{i,\alpha} = 1$ corresponding to the condition (2.8). Then the partition function is given as

$$Z = \int \mathcal{D}\bar{z}\mathcal{D}z\delta(\bar{z} \cdot z - 1)\exp\left[\int_0^\beta d\tau A(\tau)\right]$$

$$A(\tau) = -\sum_{i,\alpha} \bar{z}_{i,\alpha}\dot{z}_{i,\alpha} - \mathcal{H}(z,\bar{z}), \tag{2.10}$$

where $\mathcal{H}(z,\bar{z})$ is obtained from Eq.(2.9) by replacing $\hat{b}_{i,\alpha} \to z_{i,\alpha}$ and $\hat{b}_{i,\alpha}^\dagger \to \bar{z}_{i,\alpha}$. In order
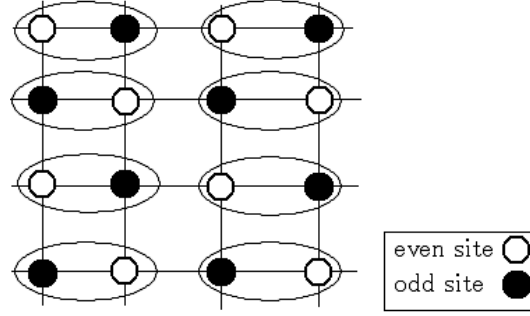
FIG. 1: CP^1 variables on odd sites are paired with their NN z_e.

to obtain the effective field theory from Eq.(2.10), we integrate out the half of the CP^{N-1} variables, e.g. those at odd sites[6]. To this end, we introduce a complete-orthogonal set of vectors in the CP^{N-1} space, $\{w_i; i = 0, 1, \cdots, N - 1\}$. Arbitrary CP^{N-1} variable z can be expanded as $z = \sum_i p_i w_i$, where p_i's are complex numbers satisfying $\sum_i |p_i|^2 = 1$. As we are interested in the case with the both exchange couplings $J_{i,j}$, $J'_{i,j} \geq 0$, dominant configurations in the path integral are given by $z_e \approx z$ at even sites and $z_o \approx \mathcal{J}\bar{z}$ at odd sites, where z is a smoothly varying CP^{N-1} field. We parameterize odd-site z_o by referring to one of its nearest-neighbor(NN) even sites z_e as (see Fig.1),

$$z_o = \sum_i p_i w_i, \tag{2.11}$$

where $w_0 = z_e$, $w_1 = \mathcal{J}\bar{z}_e$ and we do not have to specify the other w_i's. Then by the above remarks on the dominant configurations, we can assume $p_i(i \neq 1) \ll 1$ and

$$\begin{aligned} p_1 &= U(1 - \sum_{i \neq 1} |p_i|^2)^{1/2} \\ &\approx U - \frac{1}{2}U(\sum_{i \neq 1} |p_i|^2), \end{aligned} \tag{2.12}$$

where U is a U(1) variable that appears as a result of the local U(1) symmetry of the system, $z_{i,\alpha} \to e^{i\theta_i} z_{i,\alpha}$.

From Eq.(2.12),

$$z_o \approx \sum_{i \neq 1} p_i w_i + U \left(1 - \frac{1}{2} \sum_{i \neq 1} |p_i|^2 \right) w_1. \tag{2.13}$$

By substituting Eq.(2.13) into the action (2.10), we obtain

$$A(\tau) = \sum_{odd-site} (A_0 + \bar{\mathbf{p}} \cdot \mathbf{k} + \bar{\mathbf{l}} \cdot \mathbf{p} - \bar{\mathbf{p}}\mathbf{M}\mathbf{p}) + O(\mathbf{p}^4), \tag{2.14}$$

where $\mathbf{p} = (p_0, p_2, p_3, \cdots, p_{N-1})^t$. The matrix $\mathbf{M}$ is explicitly given as follows,

$$
\mathbf{M} =
\begin{pmatrix}
-4J_{o,j} - 4J'_{o,j} & 0 & \cdots & & 0 \\
0 & -4J_{o,j} & \cdots & & 0 \\
\cdot & 0 & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & 0 \\
0 & 0 & \cdots & -4J_{o,j} &
\end{pmatrix}
\tag{2.15}
$$

and the "vectors" $\mathbf{k}$ and $\bar{\mathbf{l}}$ have rather complicated form of composite of z, $\bar{z}$, w and $\bar{w}$, which are explicitly shown in the Appendix. In Eq.(2.15), $J_{o,j}$ ($J'_{o,j}$) denotes $J_{i,j}$ ($J'_{i,j}$) for $i = o$ and the NN even j (see Fig.2). As the kernel $\mathbf{M}$ has negative diagonal elements, the Gaussian integral over $\mathbf{p}$ can be safely done,

$$
\begin{aligned}
Z &= \int \mathcal{D}\bar{z}\mathcal{D}z\delta(\bar{z} \cdot z - 1) \exp\left[\int_0^\beta d\tau A(\tau)\right] \\
&= \int \mathcal{D}\bar{\mathbf{p}}\mathcal{D}\mathbf{p} \exp\left[\int_0^\beta d\tau A(\tau)\right] \\
&\approx \exp\left[\int_0^\beta d\tau A_z(\tau)\right],
\end{aligned}
\tag{2.16}
$$

$$
A_z(\tau) = \sum_{\text{odd}-\text{site}} \bar{\mathbf{l}}(\tau)\mathbf{M}^{-1}(\tau)\mathbf{k}(\tau) + \sum_{\text{odd}-\text{site}} A_0(\tau),
\tag{2.17}
$$

$$
A_0 = -\sum_j J_{o,j}(\bar{z}_j\mathcal{J}\bar{w}_1)(z_j\mathcal{J}w_1) - \sum_j J'_{o,j}(\bar{w}_1 z_j)(\bar{z}_j w_1),
\tag{2.18}
$$

where $\sum_j$ in Eq.(2.18) denotes the summation over even site j around odd site o (see Fig.2).
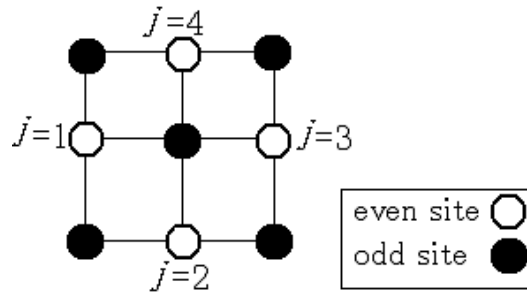

FIG. 2: z_o and its NN z_j's on even sites.

We can consider a continuum limit of the effective model (2.17), as the remaining variables z_j's can be regarded as a smoothly varying field $z(r)$ ($r_0 = \tau, r_1 = x, r_2 = y$). Hereafter we

explicitly set the exchange couplings between adjacent spins as follows,

$$J_x = J_0, \ J_y = \lambda J_0, \ J'_x = J'_0, \ J'_y = \lambda J'_0, \tag{2.19}$$

where λ is the parameter for the anisotropy between x and y directions. We take the limit $\beta = \frac{1}{T} \to \infty$, introduce explicitly the lattice spacing a and rescale the coordinates as,

$$\begin{cases} \tau \to \sqrt{2}(1+\lambda)J_0 a\tau = c\tau \\ x \to \sqrt{\frac{1+\lambda}{2}}x \\ y \to \sqrt{\frac{1+\lambda}{2\lambda}}y, \end{cases} \tag{2.20}$$

where c is the "speed of light" of the system. Then we obtain action of the effective field theory for anisotropic Sp(N) AF magnets in two dimensions,

$$S = \frac{1}{2g} \int d^3 r \left(D_\mu \bar{z} D_\mu z - \gamma(\bar{z} \mathcal{J} D_\mu \bar{z})(z \mathcal{J} D_\mu z) + \sigma(|z|^2 - 1) \right), \tag{2.21}$$

where $D_\mu = \partial_\mu - \bar{z}\partial_\mu z$, σ is the Lagrange multiplier for the CP^{N-1} constraint, γ is an increasing function of J'_0/J_0, $\gamma = J'_0/J_0 + O((J'_0/J_0)^2)$, and the coupling constant g is given by

$$g = \frac{1+\lambda}{\sqrt{2\lambda}}a. \tag{2.22}$$

From Eq.(2.22), it is obvious that the effective coupling g has the minimum at the isotropic point $\lambda = 1$. As we see in the following section, this means that the anisotropy tends to break the AF order of the ground state.

As it is shown in the Appendix, there also exists a Berry phase in the action,

$$S_B = -\frac{1}{6}\sqrt{\frac{2\lambda}{1+\lambda}}\left(1 - \frac{J'_0}{J_0}\right) \int d^3 r \epsilon_{\mu\nu\rho} D_\mu(\bar{D}_\nu \bar{z} D_\rho z). \tag{2.23}$$

The Berry phase (2.23) with the fractional coefficient depending on the anisotropy λ and J'-coupling does not suppress effects of instanton in contrast to that with an integer coefficient, i.e., S_B does not give any substantial effects on the phase structure and critical behavior[7]. For the case of the SU(2) AF magnets on 2D lattice with anisotropic couplings, this observation has been verified directly by the numerical study[8].

In the following sections, we shall study the field theory (2.21) by analytical and numerical methods.

III. PHASE STRUCTURE: ANALYTICAL STUDY

A. $1/N$ expansion: Case of small γ

The partition function of the effective field theory is given as,

$$Z = \int \mathcal{D}z\mathcal{D}\bar{z}\mathcal{D}\sigma \exp\Big(-\frac{N}{2g}\int d^3r[D_\mu\bar{z}D_\mu z - \gamma(\bar{z}\mathcal{J}D_\mu\bar{z})(z\mathcal{J}D_\mu z) + \sigma(|z|^2 - 1)]\Big), \quad (3.1)$$

where we have introduced the factor N in front of the action to perform the $1/N$-expansion in the analytical study[11]. At $\gamma = 0$, the system (3.1) has the global SU(N) symmetry, $z(r) \to Vz(r)$, $V \in$ SU(N). We first consider the case of small γ, and put the following parameterization for z, $z = z_0 + u + iv$, where $z_0 = (n_0, 0, \cdots, 0)$, $u = (0, u_2, \cdots, u_N)$ and $v = (0, v_2, \cdots, v_N)$. The fields u and v are real vectors. We have used the gauge-fixing condition $v_1 = 0$. After substituting the above parameterization to Eq.(3.1) and keeping the quadratic terms, we perform the Gaussian integration over u and v as the leading order of $1/N$,

$$Z = \int \mathcal{D}n_0\mathcal{D}\sigma\mathcal{D}u\mathcal{D}v \exp\Big[-\frac{N}{2g}\int d^3r\{(\partial_\mu u)^2 + (\partial_\mu v)^2 + \sigma(n_0^2 + u^2 + v^2 - 1)\}\Big]$$
$$= \int \mathcal{D}n_0\mathcal{D}\sigma \exp(-S_{\text{eff}}(n_0, \sigma)), \quad (3.2)$$

where

$$S_{\text{eff}}(n_0, \sigma) = (N - 1)\text{Tr}\log(-\partial_\mu^2 + \sigma) + \frac{N}{2g}\int d^3x\sigma(n_0^2 - 1). \quad (3.3)$$

As the γ-term generates only higher order terms of u and v, it does not give any effect in the leading order of $1/N$.

Gap equations are obtained as follows from $S_{\text{eff}}(n_0, \sigma)$ in Eq.(3.3),

$$\frac{\delta S_{\text{eff}}(n_0, \sigma)}{\delta\sigma} = (N - 1)\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2 + \sigma} + \frac{N}{2g}(n_0^2 - 1) = 0 \quad (3.4)$$

$$\frac{\delta S_{\text{eff}}(n_0, \sigma)}{\delta n_0} = \frac{N}{g}\sigma n_0 = 0. \quad (3.5)$$

We use the Pauli-Villars regularization with a cutoff Λ ($\sim \pi/a$) for the integral (3.4), and obtain the critical coupling g_c by putting $\sigma = n_0 = 0$,

$$\frac{1}{g_c} = \frac{1}{2\pi}\Lambda. \quad (3.6)$$

There are two phases, one for $g > g_c$,

$$\sqrt{\sigma_0} = \frac{2\pi}{N}\left(\frac{1}{g_c} - \frac{1}{g}\right), \quad n_0 = 0, \quad \sigma_0 = \langle\sigma\rangle, \quad (3.7)$$

and the other for $g < g_c$,

$$n_0^2 = 1 - \frac{g}{g_c}, \quad \sigma_0 = 0. \tag{3.8}$$

For $g < g_c$, Sp(N) symmetry is spontaneously broken and

$$\sum_{a,b} \langle (\bar{z}(r)\Gamma^{ab}z(r))(\bar{z}(r')\Gamma^{ab}z(r')) \rangle \neq 0, \quad \text{for } |r - r'| \to \infty,$$

$$\sum_a \langle (\bar{z}(r)\Gamma^a z(r))(\bar{z}(r')\Gamma^a z(r')) \rangle \neq 0, \quad \text{for } |r - r'| \to \infty. \tag{3.9}$$

In later section, the above result will be verified by the numerical study of the lattice model for the effective field theory[9].

B. Case $\gamma \approx 1$: Auxiliary fields

In this subsection, we shall consider the case $\gamma \approx 1$. It is useful to introduce two kinds of auxiliary vector fields λ_μ and ω_μ to investigate the phase structure of the model. Inserting the following identities to the partition function (3.1) (where Z_1, Z_2 are irrelevant normalization constants and will be ignored hereafter),

$$1 = Z_1 \int \mathcal{D}\lambda_\mu \exp\left[-\frac{N}{2g} \int d^3 r (\lambda_\mu - i\bar{z}\partial_\mu z)^2 \right] \tag{3.10}$$

$$1 = Z_2 \int \mathcal{D}\bar{\omega}\mathcal{D}\omega \exp\left[-\frac{N\gamma}{2g} \int d^3 r |\omega_\mu - z\mathcal{J}D_\mu z|^2 \right], \tag{3.11}$$

we obtain,

$$Z = \int \mathcal{D}z\mathcal{D}\bar{z}\mathcal{D}\sigma\mathcal{D}\lambda_\mu\mathcal{D}\bar{\omega}_\mu\mathcal{D}\omega_\mu \exp\left(-\frac{N}{2g} \int d^3 r [\bar{z}(-\partial_\mu^2 + i\lambda_\mu \overleftrightarrow{\partial}_\mu + \sigma)z \right.$$

$$\left. - z(\gamma\bar{\omega}_\mu\mathcal{J}\partial_\mu)z - \bar{z}(\gamma\omega_\mu\mathcal{J}\partial_\mu)\bar{z} + \lambda_\mu^2 + \gamma|\omega_\mu|^2 - \sigma] \right)$$

$$= \int \mathcal{D}z\mathcal{D}\bar{z}\mathcal{D}\sigma\mathcal{D}\lambda_\mu\mathcal{D}\bar{\omega}_\mu\mathcal{D}\omega_\mu \exp\left(-\frac{N}{2g} \int d^3 r [\bar{z}(-\partial_\mu^2 + i\lambda_\mu \overleftrightarrow{\partial}_\mu + \lambda_\mu^2 \right.$$

$$\left. + \gamma|\omega_\mu|^2 + \sigma)z - z(\gamma\bar{\omega}_\mu\mathcal{J}\partial_\mu)z - \bar{z}(\gamma\omega_\mu\mathcal{J}\partial_\mu)\bar{z} - \sigma] \right), \tag{3.12}$$

where $\bar{z}\lambda_\mu \overleftrightarrow{\partial}_\mu z = \lambda_\mu(\bar{z} \cdot \partial_\mu z - \partial_\mu \bar{z} \cdot z)$.

1.   Strong-coupling region

First we shall study the model (3.12) in the strong-coupling region $g > g_c$, in which $n_0 = 0$, $\langle \sigma \rangle > 0$. We put $B = \gamma\omega_\mu\mathcal{J}\partial_\mu$, $\bar{B} = \gamma\bar{\omega}_\mu\mathcal{J}\partial_\mu$ for notational simplicity. Then the

partition function is expressed as

$$
\begin{aligned}
Z = \int \mathcal{D}z\mathcal{D}\bar{z}\mathcal{D}\sigma\mathcal{D}\lambda_\mu\mathcal{D}\bar{\omega}_\mu\mathcal{D}\omega_\mu \exp\Big(-\frac{N}{2g}\int d^3r[u(-\partial_\mu^2 + \lambda_\mu^2 \\
+ \sigma + \gamma|\omega_\mu|^2 + B + \bar{B})u + u\{-\lambda_\mu\overset{\leftrightarrow}{\partial}_\mu + i(\bar{B} - B)\}v \\
+ v\{\lambda_\mu\overset{\leftrightarrow}{\partial}_\mu + i(\bar{B} - B)\}u + v\{-\partial_\mu^2 + \lambda_\mu^2 + \sigma + \gamma|\omega_\mu|^2 \\
- (B + \bar{B})\}v - \sigma]\Big),
\end{aligned}
\tag{3.13}
$$

where $z = u + iv$. As the action in Eq.(3.13) is a quadratic form of $z = u + iv$, integration over z can be done. We define

$$
\begin{aligned}
\alpha &= -\partial_\mu^2 + \lambda_\mu^2 + \gamma|\omega_\mu|^2 + \sigma, \\
\beta &= \sqrt{(\lambda_\mu\overset{\leftrightarrow}{\partial}_\mu)^2 + \gamma^2|\omega_\mu\overset{\leftrightarrow}{\partial}_\mu|^2},
\end{aligned}
\tag{3.14}
$$

and the result after the integration over z is given as

$$
Z = \int \mathcal{D}\sigma\mathcal{D}\lambda_\mu\mathcal{D}\bar{\omega}_\mu\mathcal{D}\omega_\mu \exp\Big(-\frac{N}{2}\text{Tr}\log(\alpha^2 + \beta^2) + \frac{N}{2g}\int dx^3\sigma\Big),
\tag{3.15}
$$

where

$$
\log(\alpha^2 + \beta^2) = \log\Big\{(-\partial_\mu^2 + \lambda_\mu^2 + \gamma|\omega_\mu|^2 + \sigma)^2 + (\lambda_\mu\overset{\leftrightarrow}{\partial}_\mu)^2 + \gamma^2|\omega_\mu\overset{\leftrightarrow}{\partial}_\mu|^2\Big\}.
$$

From Eq.(3.15), we shall obtain an effective action of the vector fields λ_μ and ω_μ in power of them. To this end, we use the identity

$$
\begin{aligned}
\log(\alpha^2 + \beta^2) &= 2\log(-\partial_\mu^2 + \sigma) \\
&+ \text{Tr}\Big[\frac{2}{-\partial_\mu^2 + \sigma}\lambda_\mu^2 + \frac{1}{(-\partial_\mu^2 + \sigma)^2}(\lambda_\mu\overset{\leftrightarrow}{\partial}_\mu)^2\Big] \\
&+ \text{Tr}\Big[\frac{2\gamma}{-\partial_\mu^2 + \sigma}|\omega_\mu|^2 + \frac{\gamma^2}{(-\partial_\mu^2 + \sigma)^2}|\omega_\mu\overset{\leftrightarrow}{\partial}_\mu|^2\Big] \\
&+ O(\lambda_\mu^4, \omega_\mu^4, \lambda_\mu^2\omega_\mu^2),
\end{aligned}
\tag{3.16}
$$

for positive constant σ. We explicitly evaluate the momentum integrals in Eq.(3.16),

$$
\begin{aligned}
\text{Tr}\Big[\frac{2}{-\partial_\mu^2 + \sigma}\lambda_\mu^2 &+ \frac{1}{(-\partial_\mu^2 + \sigma)^2}(\lambda_\mu\overset{\leftrightarrow}{\partial}_\mu)^2\Big] \\
&= \int \frac{d^3p}{(2\pi)^3}\lambda_\mu(p)\Pi_{\mu\nu}(p)\lambda_\nu(-p),
\end{aligned}
\tag{3.17}
$$

$$\text{Tr}\left[\frac{2\gamma}{-\partial_\mu^2 + \sigma}|\omega_\mu|^2\right] = \gamma \int \frac{d^3p}{(2\pi)^3}\bar\omega_\mu(p)\Omega_{\mu\nu}^{(1)}\omega_\nu(p), \tag{3.18}$$

$$\text{Tr}\left[\frac{\gamma^2}{(-\partial_\mu^2 + \sigma)^2}|\omega_\mu \overleftrightarrow{\partial}_\mu|^2\right] = \gamma^2 \int \frac{d^3p}{(2\pi)^3}\bar\omega_\mu(p)\Omega_{\mu\nu}^{(2)}(p)\omega_\nu(p), \tag{3.19}$$

where

$$\Pi_{\mu\nu}(p) = \int \frac{d^3q}{(2\pi)^3}\left[\frac{\delta_{\mu\nu}}{q^2 + \sigma} - \frac{(p+2q)_\mu(p+2q)_\nu}{2((p+q)^2 + \sigma)(q^2 + \sigma)}\right]$$

$$= \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Pi(p), \tag{3.20}$$

$$\Pi(p) = \frac{1}{2}(p^2 + 4\sigma)\Sigma(p) - \frac{\sqrt\sigma}{4\pi},$$

$$\Sigma(p) = \frac{1}{4\pi|p|}\tan^{-1}\sqrt{\frac{p^2}{4\sigma}},$$

$$\Omega_{\mu\nu}^{(1)} = \int \frac{d^3q}{(2\pi)^3}\frac{\delta_{\mu\nu}}{q^2 + \sigma} = \Omega^{(1)}\delta_{\mu\nu},$$

$$\Omega_{\mu\nu}^{(2)}(p) = -\int \frac{d^3q}{(2\pi)^3}\frac{(p+2q)_\mu(p+2q)_\nu}{2((p+q)^2 + \sigma)(q^2 + \sigma)}. \tag{3.21}$$

Then effective action of the vector fields λ_μ and ω_μ is obtained as follows up to the quadratic order of them,

$$\begin{aligned}
S_{\text{eff}}^{(2)}(\lambda_\mu, \omega_\mu) = {}& N \int \frac{d^3p}{(2\pi)^3}\lambda_\mu(p)\Pi_{\mu\nu}(p)\lambda_\nu(-p) \\
& + N\gamma^2 \int \frac{d^3p}{(2\pi)^3}\bar\omega_\mu(p)\Pi_{\mu\nu}(p)\omega_\nu(p) \\
& + N\gamma(1-\gamma)\int \frac{d^3p}{(2\pi)^3}\bar\omega_\mu(p)\Omega^{(1)}\omega_\nu(p) \\
& - \frac{N}{2g}\int \frac{d^3p}{(2\pi)^3}\sigma.
\end{aligned} \tag{3.22}$$

For positive σ and $p \ll 1$,

$$\Sigma(p) \approx \frac{1}{4\pi p}\sqrt{\frac{p^2}{4\sigma}} \tag{3.23}$$

$$\Pi(p) \approx \frac{1}{2}(p^2 + 4\sigma)\frac{1}{8\pi\sqrt\sigma} - \frac{\sqrt\sigma}{4\pi} = \frac{p^2}{16\pi\sqrt\sigma} \tag{3.24}$$

$$\Pi_{\mu\nu}(p) \approx \frac{1}{16\pi\sqrt\sigma}(p^2\delta_{\mu\nu} - p_\mu p_\nu), \tag{3.25}$$

and therefore the quadratic term of ω_μ becomes as follows at low momentum,

$$N \int \frac{d^3p}{(2\pi)^3}\bar\omega_\mu(p)\left[\frac{\gamma^2}{16\pi\sqrt\sigma}(p^2\delta_{\mu\nu} - p_\mu p_\nu) + \gamma(1-\gamma)\Omega^{(1)}\delta_{\mu\nu}\right]\omega_\nu(p). \tag{3.26}$$

From Eq.(3.26), it is obvious that ω_μ behaves like a massive vector field for $\gamma < 1$, for $\gamma = 1$ it becomes massless and behaves like a kind of gauge field, and finally for $\gamma > 1$ its nonvanishing condensation is expected to occur. More detailed study on the case $\gamma = 1$ will be given in later section, and it is shown there that a SU(2) gauge model really appears.

Let us study the case $\gamma > 1$ somewhat in detail for the large-N limit. Condensation of ω_μ apparently breaks the rotational symmetry of the space (or $\frac{\pi}{2}$-rotation of the square lattice) and also the U(1) gauge symmetry to Z_2[12]. Here we assume $\langle \omega_x \rangle = \omega \neq 0$ without loss of generality. We also assume that ω is real by the gauge symmetry of the system. Then the action of $z(r)$ is given as

$$S_z = \frac{N}{2g} \int d^3r [-\bar{z}\partial_\mu^2 z - \gamma\omega(\bar{z}\mathcal{J}\partial_x\bar{z}) - \gamma\omega(\bar{z}\mathcal{J}\partial_x\bar{z}) + \sigma(|z|^2 - 1)]. \tag{3.27}$$

The action S_z in Eq.(3.27) can be diagonalized by introducing field $\xi(r)$ as

$$z(r) = \frac{1}{\sqrt{2}}\{e^{-i\gamma\omega x}\xi(r) + e^{i\gamma\omega x}(i\mathcal{J}\bar{\xi}(r))\}, \tag{3.28}$$

$$S_\xi = S_z = \frac{N}{2g} \int d^3r \{\bar{\xi}(-\partial_\mu^2 + \gamma(1-\gamma)\omega^2)\xi + \sigma(\bar{\xi}\xi - 1)\}. \tag{3.29}$$

From Eq.(3.29), it is obvious that the field $\xi(r)$ acquires its mass squared $\sigma' = \sigma + \gamma(1-\gamma)\omega^2$.

From S_ξ in Eq.(3.29), we can have a gap equation and determine the critical coupling g_c. By integrating out $\xi(r)$, we obtain

$$S'_{\text{eff}}(\sigma') = N\text{Tr}\log(-\partial_\mu^2 + \sigma') - \frac{N}{2g} \int d^3x \{\sigma' - \gamma(1-\gamma)\omega^2\}, \tag{3.30}$$

and

$$\frac{S'_{\text{eff}}(\sigma')}{\delta\sigma'} = N \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + \sigma'} - \frac{N}{2g} = 0. \tag{3.31}$$

Solution to Eq.(3.31) is obtained as

$$\sqrt{\sigma'} = 2\pi \left(\frac{1}{g_c} - \frac{1}{g}\right) \tag{3.32}$$

$$\frac{1}{g_c} = \frac{\Lambda}{2\pi}. \tag{3.33}$$

The above critical value g_c is the same with that obtained for the case of small γ.

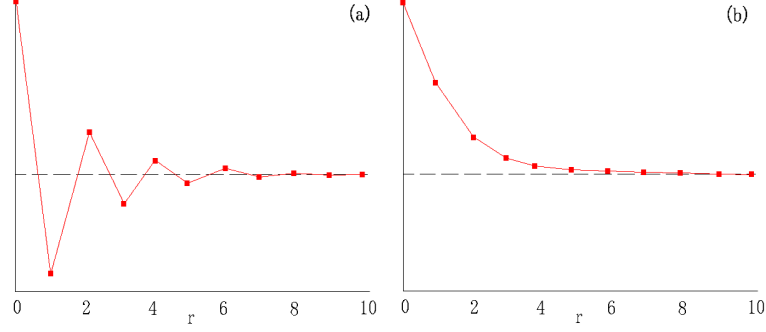In the large-N limit, spin correlations are obtained from Eq.(3.28), see Fig.3.

FIG. 3: (a)Correlation of Γ^{ab} (b)Correlation of Γ^a.

2. Weak-coupling region

We shall consider the weak coupling region $g < g_c$, in which the condensation of $z(r)$ occurs. We assume a smooth second-order phase transition from the phase of $\omega = 0$ to that of $\omega \neq 0$ as γ is increased, and estimate the critical coupling γ_c[15]. From Eq.(3.28), we divide z as $z = z_0 + \tilde{z}$, where $z_0 = (z_1, 0, \cdots, 0, z_N)$, $\tilde{z} = (0, z_2, \cdots, z_{N-1}, 0)$. Furthermore we put $\langle \sigma \rangle = 0$. Substituting the above expression of z into the action, we obtain

$$
\begin{aligned}
S &= \frac{N}{2g} \int d^3r \Big[\bar{z}(-\partial_\mu^2 + \lambda_\mu^2 + \gamma|\omega_\mu|^2 + \sigma)z \\
&\quad + i\lambda_\mu \bar{z} \overleftrightarrow{\partial}_\mu z - \bar{\omega}_\mu z \mathcal{J} \partial_\mu z - \omega_\mu \bar{z} \mathcal{J} \partial_\mu \bar{z} - \sigma \Big] \\
&= \tilde{S} + S_0,
\end{aligned}
\tag{3.34}
$$

where

$$
\tilde{S} = \frac{N}{2g} \int d^3r \Big[\bar{\tilde{z}}(-\partial_\mu^2 + \lambda_\mu^2 + \gamma|\omega_\mu|^2 + \sigma)\tilde{z} \\
+ i\lambda_\mu \bar{\tilde{z}} \overleftrightarrow{\partial}_\mu \tilde{z} - \bar{\omega}_\mu \tilde{z} \mathcal{J} \partial_\mu \tilde{z} - \omega_\mu \bar{\tilde{z}} \mathcal{J} \partial_\mu \bar{\tilde{z}} - \sigma \Big]
\tag{3.35}
$$

$$
S_0 = \frac{N}{2g} \int d^3r \Big[\bar{z}_0(-\partial_\mu^2 + \lambda_\mu^2 + \gamma|\omega_\mu|^2 + \sigma)z_0 \\
+ i\lambda_\mu \bar{z}_0 \overleftrightarrow{\partial}_\mu z_0 - \bar{\omega}_\mu z_0 \mathcal{J} \partial_\mu z_0 - \omega_\mu \bar{z}_0 \mathcal{J} \partial_\mu \bar{z}_0 \Big].
\tag{3.36}
$$

Integration over $\tilde{z}_x$ can be performed as in the previous case,

$$
\int \mathcal{D}\bar{\tilde{z}}\mathcal{D}\tilde{z}\exp(-\tilde{S})
$$

$$
= \exp\Big[-\frac{N-4}{2}\int d^3r\Big\{(-\partial_\mu^2 + \lambda_\mu^2 + \gamma|\omega_\mu|^2 + \sigma)^2
$$

$$
+ (\lambda_\mu \overleftrightarrow{\partial}_\mu)^2 + \gamma^2|\omega_\mu\partial_\mu|^2\Big\} + \frac{N}{2g}\int d^3r\,\sigma\Big]. \tag{3.37}
$$

For S_0, we substitute the following expression for z_0,

$$
z_0 = N_0 + z', \ \ N_0 = (\frac{1}{\sqrt{2}}n_0, 0, \cdots, 0, \frac{1}{\sqrt{2}}n_0),
$$

$$
z' = (z_1, 0, \cdots, 0, z_N), \tag{3.38}
$$

and obtain

$$
S_0 = \frac{N}{2g}\int d^3r\Big[\bar{z}'(-\partial_\mu^2 + \lambda_\mu^2 + \gamma|\omega_\mu|^2 + \sigma)z'
$$

$$
+ i\lambda_\mu\bar{z}'\overleftrightarrow{\partial}_\mu z' - \bar\omega_\mu z'\mathcal{J}\partial_\mu z' - \omega_\mu\bar{z}'\mathcal{J}\partial_\mu\bar{z}'\Big]
$$

$$
+ S_{\text{int}}, \tag{3.39}
$$

where

$$
S_{\text{int}} = \frac{N}{2g}\int d^3r\Big[(\lambda_\mu^2 + \gamma|\omega_\mu|^2 + \sigma)\{N_0^2 + N_0(\bar{z}' + z')\}
$$

$$
+ i\lambda_\mu N_0\partial_\mu(z' - \bar{z}') - \gamma\bar\omega_\mu N_0\mathcal{J}\partial_\mu z' - \gamma\omega_\mu N_0\mathcal{J}\partial_\mu\bar{z}'\Big]. \tag{3.40}
$$

We cannot integrate out z' exactly and therefore treat the cubic terms in S_{int} as a "perturbation" of the $1/N$-expansion[11]. Then the final expression of the effective action $S'_{\text{eff}}(\lambda_\mu, \omega_\mu)$ is given as follows up the quadratic terms of λ_μ, σ and ω_μ,

$$
S'_{\text{eff}}(\lambda_\mu, \omega_\mu) = N\lambda_\mu\Big\{\Pi_{\mu\nu} + n_0^2\Big(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)\Big\}\lambda_\mu
$$

$$
+ N\gamma^2\omega_\mu^R\Big\{\Pi_{\mu\nu} + n_0^2\Big(\frac{1}{\gamma}\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big) + \frac{1-\gamma}{\gamma}\Omega_{\mu\nu}^{(1)}\Big\}\omega_\mu^R
$$

$$
+ N\gamma^2\omega_\mu^I\Big\{\Pi_{\mu\nu} + n_0^2\Big(\frac{1}{\gamma}\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big) + \frac{1-\gamma}{\gamma}\Omega_{\mu\nu}^{(1)}\Big\}\omega_\mu^I
$$

$$
+ N\sigma\Big(\frac{1}{2}\Sigma(p) + n_0^2\Big)\sigma - \frac{N}{2g}\sigma, \tag{3.41}
$$

where $\omega_\mu = \omega_\mu^R + i\omega_\mu^I$.

It is obvious that for $n_0 \neq 0$ case, only the transverse modes of ω_μ (i.e., $\sum_\mu p_\mu \omega_\mu = 0$) can survive at low momentum. Furthermore its mass is renormalized by $n_0 \neq 0$ as $\gamma[n_0^2 + (1-\gamma)\Omega^{(1)}]$. Then for $\gamma > \gamma_c = 1 + \frac{n_0^2(g)}{\Omega^{(1)}}$, $\xi(r)$, instead of $z(r)$, condenses and nontrivial correlations of the "spin operators", $\hat{\Gamma}^{ab}$, appear as a result of $\langle \omega_\mu \rangle \neq 0$. In this phase,

$$
\begin{aligned}
\langle \hat{\Gamma}^{ab}(r) \rangle &= \langle \bar{z}(r) \Gamma^{ab} z(r) \rangle \\
&= n_1^{ab} \cos(2\gamma\omega x) + n_2^{ab} \sin(2\gamma\omega x),
\end{aligned}
\tag{3.42}
$$

where $n_1^{ab} = \mathrm{Re}[\langle \xi \rangle \mathcal{J} \Gamma^{ab} \langle \xi \rangle]$, $n_2^{ab} = \mathrm{Im}[\langle \xi \rangle \mathcal{J} \Gamma^{ab} \langle \xi \rangle]$. On the other hand, $\langle \hat{\Gamma}^a(r) \rangle = \langle \bar{\xi} \rangle \Gamma^a \langle \xi \rangle$.

IV.   SU(2) GAUGE THEORY AT $\gamma = 1$

In the previous section, we found that the composite vector field ω_μ behaves like a massless gauge field at $\gamma = 1$, and for $g < g_c$ it acquires mass squared propotional to $n_0^2 = \langle \bar{z} \cdot z \rangle$ as a result of the Higgs mechanism. In this section, we shall explicitly show that the three real vector fields $(\lambda_\mu, \omega_\mu^R, \omega_\mu^I)$ form a SU(2) gauge field minimally coupled with z_x at $\gamma = 1$. This is another example of the symmetry-enhancement phenomenon, i.e., emergent symmetry at low energies.

We start with the action $S_{\gamma=1}$ in Eq.(3.12),

$$
S_{\gamma=1} = \frac{N}{2g} \int d^3r \left[\bar{z}(-\partial_\mu^2 + i\lambda_\mu \overleftrightarrow{\partial}_\mu + \lambda_\mu^2 + |\omega_\mu|^2 + \sigma)z - z(\bar{\omega}_\mu \mathcal{J} \partial_\mu)z - \bar{z}(\omega_\mu \mathcal{J} \partial_\mu)\bar{z} - \sigma \right],
\tag{4.1}
$$

where we have put $\gamma = 1$. Hereafter we explicitly consider the CP^3 case but generalization to an arbitrary N is straightforward. We first redefine the CP^3 field $Z(r)$ from the original $z(r)$ as follows, $Z(r) = (z_1(r), z_2(r), \bar{z}_4(r), -\bar{z}_3(r))^t$. It is easily verified that $Z(r)$ is a CP^3 field, $\sum_{i=1}^4 |Z_i(r)|^2 = 1$.

It is straightfward to verify the following equation,

$$
\bar{z} \overleftrightarrow{\partial}_\mu z = \bar{Z} \Sigma_3 \overleftrightarrow{\partial}_\mu Z,
\tag{4.2}
$$

where

$$\Sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.3}$$

Similarly

$$\bar{\omega}_\mu(z\mathcal{J}\partial_\mu)z + \omega_\mu(\bar{z}\mathcal{J}\partial_\mu)\bar{z} = i\omega_\mu^R(\bar{Z}\Sigma_2 \overleftrightarrow{\partial}_\mu Z) + i\omega_\mu^I(\bar{Z}\Sigma_1 \overleftrightarrow{\partial}_\mu Z), \tag{4.4}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.5}$$

It is obvious that $\Sigma_i(i = 1, 2, 3)$ satisfy the SU(2) algebra. Let us define SU(2) gauge field $\vec{\mathcal{A}}_\mu$ as $\vec{\mathcal{A}}_\mu = (\lambda_\mu^1, \lambda_\mu^2, \lambda_\mu^3) = (\omega_\mu^I, \omega_\mu^R, \lambda_\mu)$, then the Lagrangian (4.1) can be rewritten as follows,

$$S_{\gamma=1} = \frac{N}{2g} \int d^3r \left[|(\partial_\mu + i\vec{\Sigma} \cdot \vec{\mathcal{A}}_\mu)Z|^2 + \sigma|Z|^2 - \sigma \right]. \tag{4.6}$$

Lattice gauge model corresponding to the above SU(2) gauge theory (4.6) is under study and result will be reported in a future publication. However phase structure of the system can be inferred by qualitative discussion. As in the usual CP^{N-1} model coupled with the U(1) gauge field, there exists a phase transition that separates ordered and disordered phases. However, as the SU(2) gauge field fluctuates the hopping of the spinon $Z(r)$ more strongly than the U(1) gauge field, the critical coupling $g_c(\gamma = 1)$ is expected to be smaller than $g_c(\gamma = 0)$. From this consideration, we expect that the critical coupling $g_c(\gamma)$ is a decreasing function of γ for $\gamma < \gamma_c$, though in the previous discussion for small $\gamma \ll 1$ by the $1/N$-expansion we do not find any γ dependence of g_c. As γ exceeds γ_c, the condensation of ω_μ tends to occur and fluctuations of ω_μ are suppressed. Moreover the original U(1) gauge symmetry reduces to Z_2 gauge symmetry by the Anderson-Higgs mechanism and fluctuations

of λ_μ are also suppressed. Then $g_c(\gamma)$ starts to increase at $\gamma = \gamma_c$. The above expectation will be confirmed by the numerical study of lattice-gauge model in the following section. Expected phase diagram is show in the $(\gamma - g)-$ plane in Fig.4.
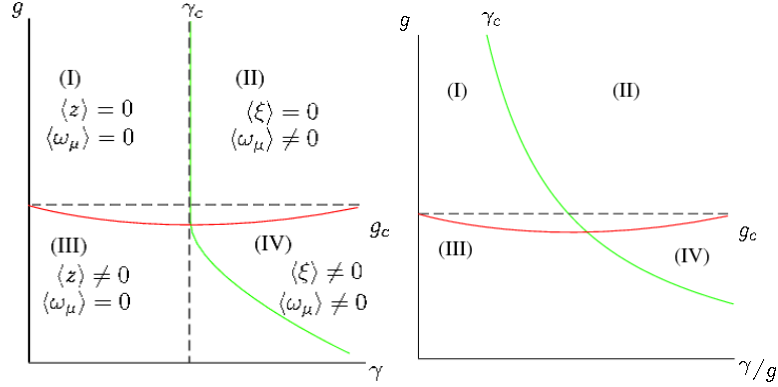

FIG. 4: Phase diagram in $(\gamma - g)$ and $(\gamma/g - g)-$ planes.

In the following section, we shall introduce a lattice model for the effective field theory with general value of γ, and study it by means of MC simulations.
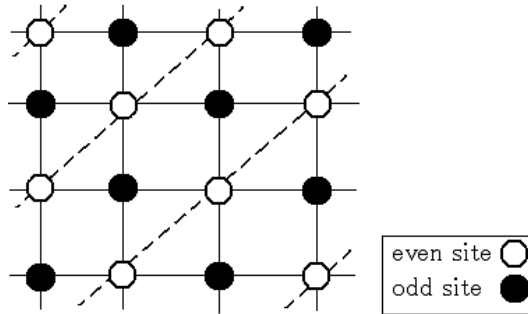
FIG. 5: Interaction terms in A_L in Eq.(5.1).

V.   NUMERICAL STUDY

In this section we formulate the effective field theory (3.1) on a cubic lattice, and investigate its phase structure by means of the MC simulations. We explicitly consider the Sp(4)
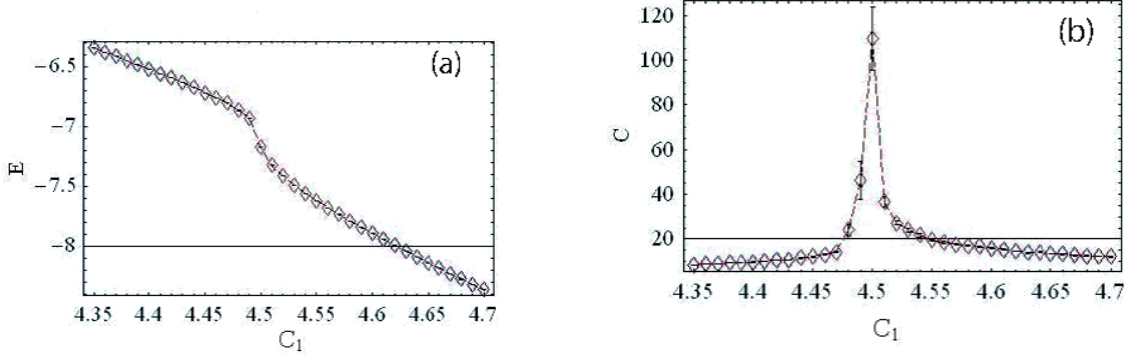
FIG. 6: (a)Enegry E as a function of c_1 for $c_2 = c_3 = 0$. There exists a discontinuity at $c_1 = 4.5$. (b)"Specific heat" C as a function of c_1 for $c_2 = c_3 = 0$. There exists a sharp peak at $c_1 = 4.5$. System size $L = 20$.

model. Action of the lattice model A_L is given as follows,

$$A_L = c_1 \sum_{r,\mu} \bar{z}_{r+\mu} U_{r\mu} z_r + \text{c.c.}$$
$$+ c_2 \sum_{r,\mu,\nu} U_{r\mu} U_{r+\mu,\nu} \bar{U}_{r+\nu,\mu} \bar{U}_{r\nu} + \text{c.c.}$$
$$+ c_3 \sum_{r,\mu} |z_r \mathcal{J} z_{r+\mu}|^2 + c_4 \sum_{r,\mu} |z_r \mathcal{J} z_{r+1+2}|^2, \tag{5.1}$$

where r denotes the cubic lattice site, $\mu = (0, 1, 2)$ is the direction index and it also denotes the unit vector in the μ-direction. Field z_r's are CP^3 variables and $U_{r\mu}$ is a U(1) gauge field defined on link (r, μ), $U_{r\mu} \sim e^{i\lambda_\mu(r)}$. The parameters $c_1 \propto 1/g$, $c_3 \propto \gamma/g$ and the c_2-term is the lattice Maxwell term (the so-called Wilson term) corresponding to $(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$ in the continuum spacetime[15]. We also added the c_4-term on the diagonal lines in the 2D layers that represents the exchange couplings between spins on even sites of the original lattice. See Fig.5. The partition function Z_L is given as

$$Z_L = \int [DU][DzD\bar{z}]_{\text{CP}} \, e^{A_L}, \tag{5.2}$$

where $[DzD\bar{z}]_{\text{CP}}$ denotes the integration over CP^3 variables, and we use a specific parameterization for them in the MC simulations. For the MC simulations, we used the standard Metropolis algorithm of local update. The typical statistics was 10^5 MC steps per sample, and the averages and errors were estimated over ten samples. The typical acceptance ratio was about 50%. We also used multi-histogram methods to obtain reliable results near the phase transition point[14].
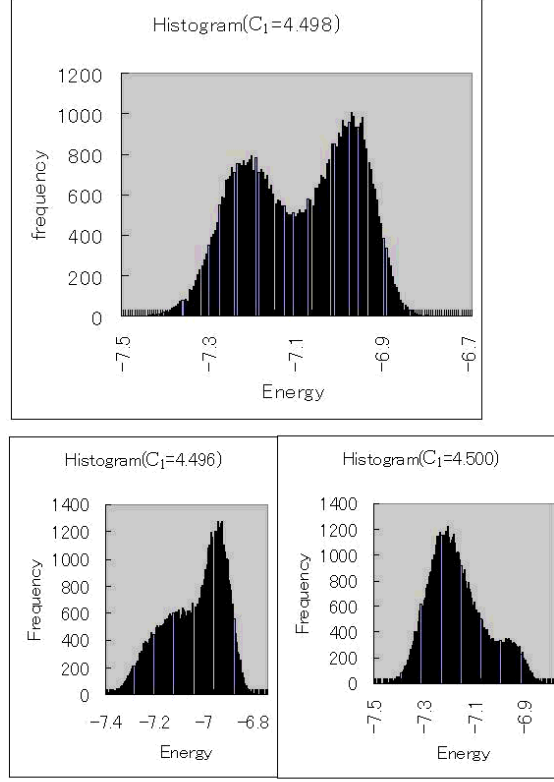
FIG. 7: Enegry distribution $N[E]$ for the pure CP^3 model. At $c_1 = 4.498$, $N(E)$ has double-peak shape, whereas it has single peak at $c_1 = 4.496$ and 4.500. System size $L = 20$.

We first consider the case of the pure CP^3 model with $c_2 = c_3 = c_4 = 0$, which corresponds to the anisotropic SU(4) AF magnet[10]. We calculate the "internal energy" $E = \langle A_L \rangle / L^3$ and the "specific heat" $C = \langle (A_L - E)^2 \rangle / L^3$ to study phase structure, where L^3 is the lattice size and we impose the periodic boundary condition in moose of calculations. In Fig.6, we show E and C as a function of c_1. It is obvious that E has a discontinuity at $c_1 \simeq 4.5$ and C has a very large peak at $c_1 \simeq 4.5$, which indicates a first-order phase transition. In order to verify this observation, we calculated density of states $N[E]$ that is defined as

$$N[E] = \int [DU][DzD\bar{z}]_{\mathrm{CP}} \, \delta(A_L - E) \, e^{A_L}. \tag{5.3}$$

Result in Fig.7 shows that $N[E]$ has a double-peak shape at $c_1 = 4.498$, whereas it has a single peak at the other couplings. This confirms the existence of the first-order phase transition in the CP^3 model, though corresponding phase transitions in the CP^1 and CP^2 models are of second order. Study of the correlation functions of the "spin" operators given
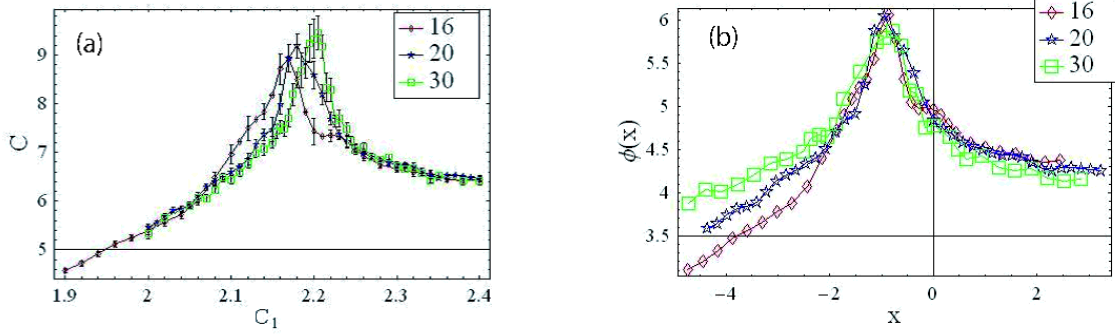
FIG. 8: (a)C for $c_2 = 2.0$, $c_3 = 0$. System size is $L = 16$, 20, 30. (b)Scaling function $\phi(x)$ in finite-size scaling (5.4). Critical exponents and critical coupling are estimated as $\nu = 0.8$, $\sigma = 0.11$ and $c_{1c} = 2.23$.

later on shows that phase transition from "Néel" to paramagnetic states takes place at $c_1 = 4.498$.

CP^{N-1} model in the 3D spacetime was studied by the $1/N$-expansion and it was suggested that there existed a second-order phase transition from ordered to disordered phases as the coupling constant is increased. However the present investigation by means of the MC simulations shows that the order of the phase transition varies as a function of the parameter N. Similar phenomenon was recently observed some related model, e.g. multi-Higgs U(1) gauge model in 3D. We also studied finite but small c_2 cases and found that the phase transition is still of first-order. However at intermediate value of c_2, C exhibits the finite-size scaling (see Fig.8), i.e., the data of $C(c_1, L)$ for system size L and $c_2 = 2.0$ can be fit as follows with a scaling function $\phi(x)$[16],

$$C(c_1, L) = L^{\sigma/\nu}\phi(L^{1/\nu}\epsilon), \quad \epsilon = (c_1 - c_{1c})/c_{1c}, \tag{5.4}$$

where c_{1c} is the critical coupling at infinite system size and estimated as $c_{1c} = 2.23$. This fact means that the phase transition becomes of second-order as the value of c_2 is increased.

Let us turn on the c_3-term and see how location of the phase transition varies. We studied the system by varying value of c_1 with fixed c_3, and found clear signal of phase transitions, see, e.g. C for $c_2 = 2.0$, $c_3 = 1.0$ in Fig.9. We also investigated phase structure of the system with c_1 fixed and c_3 varied, and found that there is another phase transition line. See Fig.9.
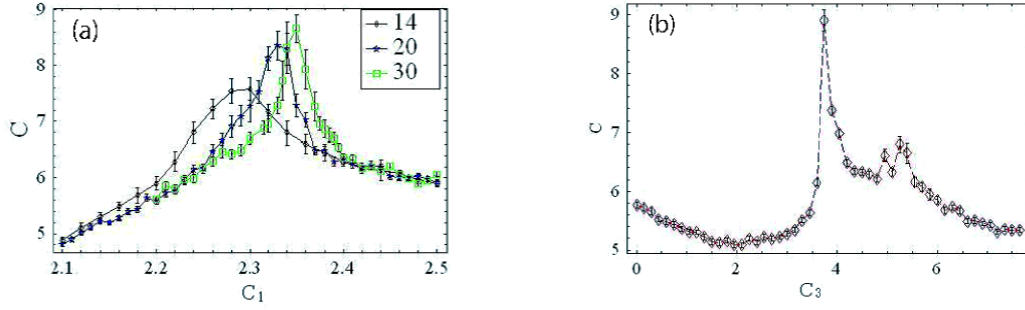
FIG. 9: (a)C for $c_2 = 2.0$, $c_3 = 1.0$. System size is $L = 14$, 20, 30. Critical coupling is estimated at $c_1 = 2.35$. (b)C for $c_1 = 3.0$, $c_2 = 2.0$. System size is $L = 18$. There are two phase transitions at $c_3 = 3.7$ and 5.2.
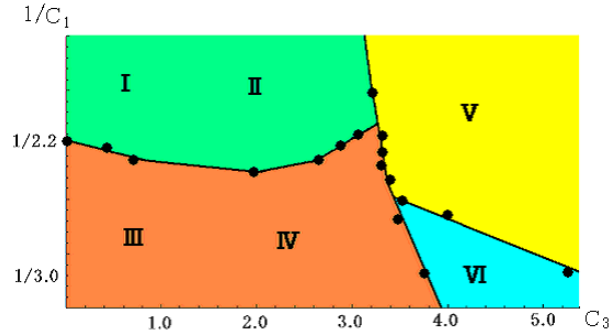

FIG. 10: Phase diagram in $(c_3 - \frac{1}{c_1})$ plane for $c_2 = 2.0$. Solid lines are phase transition lines obtained by measurement of E and C. Dots denote phase transition points actually observed by measurement. "Spin" correlation functions exhibit different behavior in regions $I \sim VI$. Similar phase structure is obtained for $c_2 = 0$.

Obtained phase diagram in the $(c_3 - \frac{1}{c_1})$ plane is shown in Fig.10. There are four phases, which are identified by the measurement of E and C. We also investigated the behavior of the correlation function of the "spin" operators,

$$G_n(r) = \sum_a \langle \Gamma^a_{r'+r} \Gamma^a_{r'} \rangle, \quad G_s(r) = \sum_{ab} \langle \Gamma^{ab}_{r'+r} \Gamma^{ab}_{r'} \rangle. \tag{5.5}$$

As we expect that a phase transition to a "spiral state" takes place as the parameters are increased, we take the free boundary condition in the two spatial directions. We found
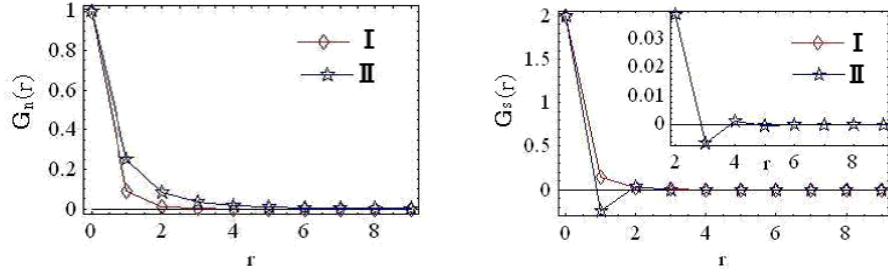
FIG. 11: Spin correlation functions in I and II in phase diagram Fig.10. Both $G_n(r)$ and $G_s(r)$ have no long-range order in I and II, however in II $G_s(r)$ exhibits a short-range spiral order.
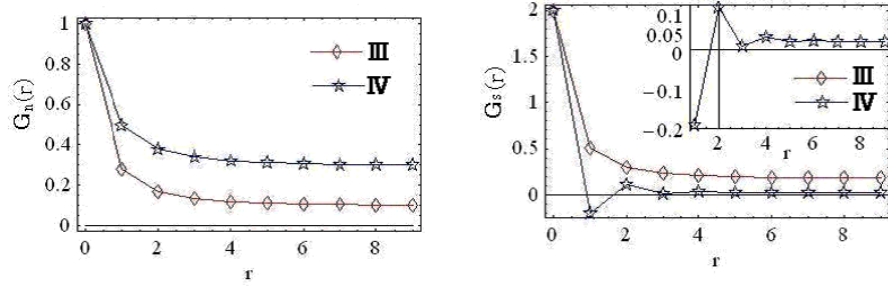

FIG. 12: Spin correlation functions in III and IV in phase diagram Fig.10. $G_n(r)$ has long-range order both in III and IV. In IV, $G_s(r)$ exhibits a long-range spiral order, whereas it has usual long-range order in III.

that the correlators exhibit different behavior in the six "regions" $I \sim VI$ shown in Fig.10. It is obvious that not only simple ferromagnetic(FM) correlation, antiferromagnetic(AF) correlations appear in these correlators. For example in the regions I and II (III and IV), the correlation of the nematic order $G_n(r)$ exhibits the same behavior, but the spin correlator $G_s(r)$ behaves differently in I and II (III and IV), see Fig.11 (Fig.12). We have observed no phase boundary between the I and II (III and IV) regions on which the internal energy E and specific heat C exhibit anomalous behavior. However from the result of the spin correlation function, $G_s(r)$, we expect that the phases $\langle \omega_\mu \rangle \neq 0$, $\langle \xi(r) \rangle = 0$ and $\langle \omega_\mu \rangle \neq 0$, $\langle \xi(r) \rangle \neq 0$ are realized in the regions II and IV, respectively. On the other hand, there are no phases in the effective field theory that correspond to the phases V and VI of

the FM long-range order in the lattice model.

Finally we turn on the c_4-term in addition to the c_3-term. Numerical study of the system was performed along the line $c_3 = c_4$ with $c_1 = 6.0$ and $c_2 = 0$ in the phase diagram. Observed C as a function of $c_3 = c_4$ is shown in Fig.14. There are two phase transition, one at $c_3 = c_4 \simeq 0.7$ and the other at $c_3 = c_4 \simeq 6.0$. In order to understand these phase transitions, we calculated the spin correlation functions in each phase, which are shown in Fig.15. From the result, it is obvious that the AF order disappears first and then at the second transition the spiral state appears. This state corresponds to the state with nonvanishing $\langle \xi(r) \rangle \neq 0$ studied in the previous sections.

VI. CONCLUSION

In the present paper, we have studied the Sp(N) AF Heisenberg model in 2D that is expected to be realized in cold atom system in an optical lattice. We focus on the ground state structure of the system and derived the effective field theory for the system, which is a kind of extension of the CP^{N-1} nonlinear σ-model. Then we studied the phase structure and critical behavior by using the $1/N$-expansion. We found that the spatial anisotropy induces a phase transition from the ordered state to paramagnetic state. As the explicit breaking of the SU(N) symmetry increases, the ground state exhibits a spiral order of the ad joint representation of Sp(N) group. This state also has a nematic order of vector representation of the Sp(N) group.

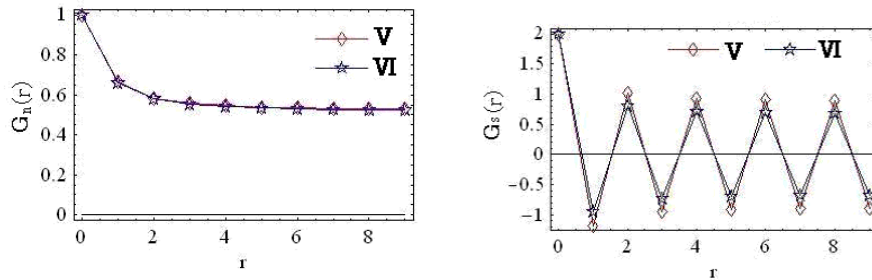We also introduced a lattice gauge-model counterpart of the effective field theory, and

FIG. 13: Spin correlation functions in V and VI in phase diagram Fig.10. Both $G_n(r)$ and $G_s(r)$ exhibit the same behavior in V and VI.
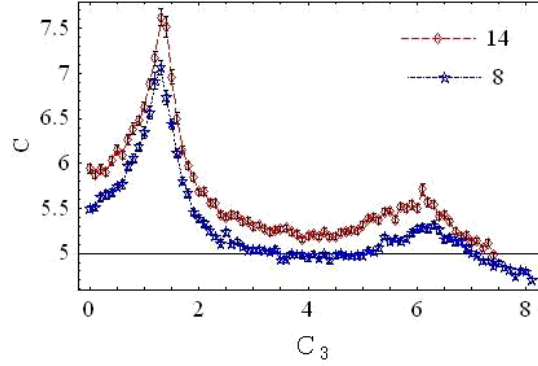
FIG. 14: Specific heat on line $c_3 = c_4$ with $c_1 = 6$, $c_2 = 0$. Result indicates two phase transitions on the line.
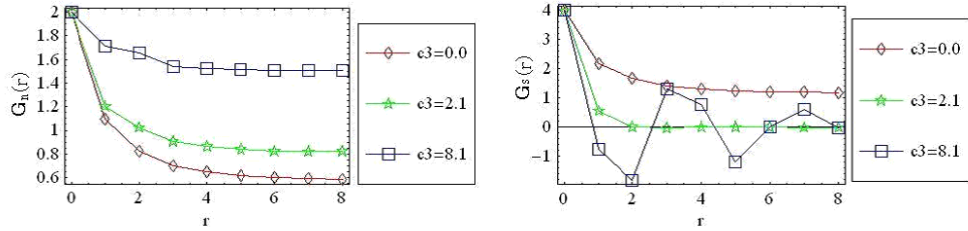

FIG. 15: Spin correlation functions for $c_1 = 6.0$ and $c_2 = 0$.

studied its phase structure by means of the MC simulations. We found a similar phase diagram with that of the field theory, but order of the phase transitions is different in two systems.

In Ref.[17], finite-temperature phase diagram of Sp(N) model is studied by using a Ginzburg-Landau theory in terms of gauge-invariant spin fields, $\phi^{ab} = z^\dagger \Gamma^{ab} z$, $\phi^a = z^\dagger \Gamma^{ab} z$. There quantum phase transition of Sp(4) spin system in 3D stacked square lattice is also discussed. Phase transition from CP^3 Néel ordered state to photon liquid state is predicted. In the (3+1)D counterpart of the lattice model (5.1), it is expected that deconfined photon phase exists for sufficiently large c_2. U(1) gauge theory of CP^1 field in (3+1)D was studied previously[18], and it was found that phase transition from CP^1 Néel ordered state to photon liquid state is of second order.

It is interesting to study effects of hole doping to the Sp(N) AF magnets and investigate

how the long-range orders are broken and if a new phase with hole-pair condensation appears. This system is an extension of the t-J model for high-temperature superconductivity. This problem is under study and we hope that results will be reported in a future publication.

Acknowledgment

Appendix A: Derivation of the effective field theory

In Sec.2, we derived the effective field theory of the Sp(N) AF Heisenberg model. In the present appendix, we give some details of the derivation.

In Eq.(2.14), $A(\tau)$ is expanded in powers of $\mathbf{p}$ and $\bar{\mathbf{p}}$. Vectors $\mathbf{k}$ and $\bar{\mathbf{l}}$ are explicitly given as follows,

$$
\mathbf{k} = \begin{pmatrix}
-U(\bar{w}_0 \dot{w}_1) - \sum_j J_{o,j}(\bar{z}_j \mathcal{J} \bar{w}_0)(z_j \mathcal{J} w_1)U - \sum_j J'_{o,j}(\bar{w}_0 z_j)(\bar{z}_j w_1)U \\
-U(\bar{w}_2 \dot{w}_1) - \sum_j J_{o,j}(\bar{z}_j \mathcal{J} \bar{w}_2)(z_j \mathcal{J} w_1)U - \sum_j J'_{o,j}(\bar{w}_2 z_j)(\bar{z}_j w_1)U \\
-U(\bar{w}_3 \dot{w}_1) - \sum_j J_{o,j}(\bar{z}_j \mathcal{J} \bar{w}_3)(z_j \mathcal{J} w_1)U - \sum_j J'_{o,j}(\bar{w}_3 z_j)(\bar{z}_j w_1)U
\end{pmatrix}
\tag{A.1}
$$

$$
\bar{\mathbf{l}} = \begin{pmatrix}
-\bar{U}(\bar{w}_1 \dot{w}_0) - \sum_j J_{o,j}(z_j \mathcal{J} w_0)(\bar{z}_j \mathcal{J} \bar{w}_1)\bar{U} - \sum_j J'_{o,j}(\bar{z}_j w_0)(\bar{w}_1 z_j)\bar{U} \\
-\bar{U}(\bar{w}_1 \dot{w}_2) - \sum_j J_{o,j}(z_j \mathcal{J} w_2)(\bar{z}_j \mathcal{J} \bar{w}_1)\bar{U} - \sum_j J'_{o,j}(\bar{z}_j w_2)(\bar{w}_1 z_j)\bar{U} \\
-\bar{U}(\bar{w}_1 \dot{w}_3) - \sum_j J_{o,j}(z_j \mathcal{J} w_3)(\bar{z}_j \mathcal{J} \bar{w}_1)\bar{U} - \sum_j J'_{o,j}(\bar{z}_j w_3)(\bar{w}_1 z_j)\bar{U}
\end{pmatrix}
\tag{A.2}
$$

After integrating out $\mathbf{p}$ and $\bar{\mathbf{p}}$, we obtain Eq.(2.17). For $J'_{o,j} \ll J_{oj}$, the inverse of the matrix $\mathbf{M}$ can be approximated as

$$
\begin{aligned}
\mathbf{M}^{-1} &= \begin{pmatrix}
-(4J_{o,j} + 4J'_{o,j})^{-1} & 0 & 0 \\
0 & -(4J_{o,j})^{-1} & 0 \\
0 & 0 & -(4J_{o,j})^{-1}
\end{pmatrix} \\
&\approx \begin{pmatrix}
-\frac{1}{4J_{o,j}} + \frac{J'_{o,j}}{4J^2_{o,j}} & 0 & 0 \\
0 & -\frac{1}{4J_{o,j}} & 0 \\
0 & 0 & -\frac{1}{4J_{o,j}}
\end{pmatrix}
\end{aligned}
\tag{A.3}
$$

and

$$A_z(\tau) = \sum_{\text{odd}} \left[- \sum_k{}' \frac{1}{2J_k} (\bar{w}_k \dot{w}_1)(\bar{w}_1 \dot{w}_k) \right.$$

$$- J_{o,i} \sum_k{}' \frac{1}{2J_k} \sum_i \{ (\bar{z}_i \mathcal{J} \bar{w}_k)(z_i \mathcal{J} w_1)(\bar{w}_1 \dot{w}_k) + (z_i \mathcal{J} w_k)(\bar{z}_i \mathcal{J} \bar{w}_1)(\bar{w}_k \dot{w}_1) \}$$

$$- J_{o,i}^2 \sum_k{}' \frac{1}{2J_k} \sum_{i,j} (\bar{z}_i \mathcal{J} \bar{w}_k)(z_i \mathcal{J} w_1)(z_j \mathcal{J} w_k)(\bar{z}_j \mathcal{J} \bar{w}_1)$$

$$- J_{o,i}' \sum_k{}' \frac{1}{2J_k} \sum_i \{ (\bar{w}_k z_i)(\bar{z}_i w_1)(\bar{w}_1 \dot{w}_k) + (\bar{z}_i \bar{w}_k)(\bar{w}_1 z_i)(\bar{w}_k \dot{w}_1) \}$$

$$- J_{o,i} J_{o,i}' \sum_k{}' \frac{1}{2J_k} \sum_{i,j} \{ (\bar{z}_i \mathcal{J} \bar{w}_k)(z_i \mathcal{J} w_1)(\bar{z}_j w_k)(\bar{w}_1 z_j)$$

$$+ (z_i \mathcal{J} w_k)(\bar{z}_i \mathcal{J} \bar{w}_1)(\bar{w}_k z_j)(\bar{z}_j w_1) \}$$

$$- (J_{o,i}')^2 \sum_k{}' \frac{1}{2J_k} \sum_{i,j} (\bar{w}_k z_i)(\bar{z}_i w_1)(\bar{z}_j w_k)(\bar{w}_1 z_j)$$

$$\left. - J_{o,i} \sum_i (\bar{z}_i \mathcal{J} \bar{w}_1)(z_i \mathcal{J} w_1) - J_{o,i}' \sum_i (\bar{w}_1 z_i)(\bar{z}_i w_1) \right], \tag{A.4}$$

where $\sum_k{}'$ denotes the sum over $k = 0, 2, 3$, and $J_0 \approx 4J_{o,i} + 4J_{o,i}'AJ_2 = J_3 = 4J_{o,i}$. By using identity such as

$$(\bar{z}_i z_j) = (\bar{z}_i z)(\bar{z} z_j) + a^2 (\bar{D}_i \bar{z} D_j z),$$

$$D_\mu = \partial_\mu + iA_\mu, \quad A_\mu = i\bar{z}\partial_\mu z, \tag{A.5}$$

where a is the lattice spacing, and the completeness of $\{w_k\}$,

$$\sum_{k=0}^3 w_{k\alpha} \bar{w}_{k\beta} = \delta_{\alpha\beta}, \tag{A.6}$$

we obtain

$$
\begin{aligned}
A_z(\tau) = \sum_{\text{odd}} \Bigg[& \frac{1}{2Jd} \bar{D}_\tau \bar{z} D_\tau z - \frac{Ja^2}{2d} \sum_{i,j} \bar{D}_i \bar{z} D_j z \\
& - \frac{a}{2d} \sum_j (\bar{D}_\tau \bar{z} D_j z - \bar{D}_j \bar{z} D_\tau z) + Ja^2 \sum_j \bar{D}_j \bar{z} D_j z - 2Jd \\
& - \frac{J'}{2J^2 d} (\bar{z} \mathcal{J} \bar{D}_\tau \bar{z})(z \mathcal{J} D_\tau z) + \frac{J'a^2}{2d} \sum_{i,j} (\bar{z} \mathcal{J} \bar{D}_i \bar{z})(z \mathcal{J} D_j z) \\
& - J'a^2 \sum_j (\bar{z} \mathcal{J} \bar{D}_j \bar{z})(z \mathcal{J} D_j z) \\
& - \frac{J'}{2Jd} \sum_j \{ (\bar{z} \mathcal{J} \bar{D}_j \bar{z})(z \mathcal{J} D_\tau z) - (\bar{z} \mathcal{J} \bar{D}_\tau \bar{z})(z \mathcal{J} D_j z) \} \\
& + \frac{J'}{2Jd} \sum_j \{ (\bar{z} \mathcal{J} \bar{D}_j \bar{z})(z_j \mathcal{J} D_\tau z) - (z \mathcal{J} D_j z)(\bar{z}_j \mathcal{J} \bar{D}_\tau \bar{z}) \} \Bigg] .
\end{aligned} \tag{A.7}
$$

From Fig.2, by substituting

$$
\begin{cases}
D_1 = (-)^y D_x + D_y \\
D_2 = 2(-)^y D_x \\
D_3 = (-)^y D_x - D_y,
\end{cases} \tag{A.8}
$$

the effective action S_{E} defined by

$$
S_{\text{E}} = \int_0^\beta d\tau A_z(\tau) \tag{A.9}
$$

is obtained as

$$
\begin{aligned}
S_{\text{E}} = \int_0^\beta d\tau \Bigg[& \frac{1}{4J} \bar{D}_\tau \bar{z} D_\tau z + 2Ja^2 \sum_{\mu=x,y} \bar{D}_\mu \bar{z} D_\mu z - a(-)^y (\bar{D}_\tau \bar{z} D_x z - \bar{D}_x \bar{z} D_\tau z) \\
& - \frac{J'}{4J^2} (\bar{z} \mathcal{J} \bar{D}_\tau \bar{z})(z \mathcal{J} D_\tau z) - 2J'a^2 \sum_{\mu=x,y} (\bar{z} \mathcal{J} \bar{D}_\mu \bar{z})(z \mathcal{J} D_\mu z) \\
& - \frac{J'}{J} (-)^y \{ (\bar{z} \mathcal{J} \bar{D}_x \bar{z})(z \mathcal{J} D_\tau z) - (\bar{z} \mathcal{J} \bar{D}_\tau \bar{z})(z \mathcal{J} D_x z) \} \\
& + \frac{J'}{4J} \sum_j \{ (\bar{z} \mathcal{J} \bar{D}_j \bar{z})(z_j \mathcal{J} D_\tau z) - (z \mathcal{J} D_j z)(\bar{z}_j \mathcal{J} \bar{D}_\tau \bar{z}) \} \Bigg] .
\end{aligned} \tag{A.10}
$$

By the rescaling (2.20),

$$S = \frac{\sqrt{2\lambda}}{2a(1+\lambda)} \int d^3r \Bigg[\sum_{\mu=\tau,x,y} \bar{D}_\mu \bar{z} D_\mu z - \sqrt{1+\lambda} a \partial_y (\bar{D}_\tau \bar{z} D_x z - \bar{D}_x \bar{z} D_\tau z)$$

$$- \frac{J_0'}{J_0} \sum_{\mu=\tau,x,y} (\bar{z} \mathcal{J} \bar{D}_\mu \bar{z})(z \mathcal{J} D_\mu z) - \frac{J_0'}{J_0} \sqrt{1+\lambda} a \partial_y (\bar{D}_x \bar{z} D_\tau z - \bar{D}_\tau \bar{z} D_x z)$$

$$+ \sum_j \frac{J_j'}{4J_0} \sqrt{1+\lambda} \{ (\bar{z} \mathcal{J} \bar{D}_j \bar{z})(z_j \mathcal{J} D_\tau z) - (z \mathcal{J} D_j z)(\bar{z}_j \mathcal{J} \bar{D}_\tau \bar{z}) \} \Bigg]. \tag{A.11}$$

The last term in Eq.(A.11) is rewritten as

$$\sum_j \frac{J_j'}{4J_0} \sqrt{1+\lambda} \{ (\bar{z} \mathcal{J} \bar{D}_j \bar{z})(z_j \mathcal{J} D_\tau z) - (z \mathcal{J} D_j z)(\bar{z}_j \mathcal{J} \bar{D}_\tau \bar{z}) \}$$

$$\rightarrow \sum_j \frac{J_j'}{4J_0} \sqrt{1+\lambda} \{ (\bar{z} \mathcal{J} \bar{D}_j \bar{z})(z \mathcal{J} D_\tau z) - (z \mathcal{J} D_j z)(\bar{z} \mathcal{J} \bar{D}_\tau \bar{z}) \}$$

$$= \frac{J_0'}{J_0} \sqrt{1+\lambda} (\bar{D}_x \bar{z} D_\tau z - \bar{D}_\tau \bar{z} D_x z), \tag{A.12}$$

then,

$$S_{\mathrm{E}} = \frac{\sqrt{2\lambda}}{2a(1+\lambda)} \int d^3r \Bigg[\sum_{\mu=\tau,x,y} \bar{D}_\mu \bar{z} D_\mu z - \frac{J_0'}{J_0} \sum_{\mu=\tau,x,y} (\bar{z} \mathcal{J} \bar{D}_\mu \bar{z})(z \mathcal{J} D_\mu z) \Bigg] + S_B, \tag{A.13}$$

where S_B is the Berry phase,

$$S_B = -\frac{1}{2} \sqrt{\frac{2\lambda}{1+\lambda}} \Big(1 - \frac{J_0'}{J_0}\Big) \int d^3r D_y (\bar{D}_\tau \bar{z} D_x z - \bar{D}_x \bar{z} D_\tau z). \tag{A.14}$$

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